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DEPARTMENT OF THE ARMY
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2554

Translation No. T-689-2

Author: R. Gans, Physikalisches Institut, Tubingen.

Title: How do rods and discs fall in a friction-producing liquid? (Wie fallen Stabe und Scheiben in einer reibenden Flussigkeit?)

Journal: Sitzungsb. d. Math.-Phys. Kl., 1911, pp. 191-203.

August 1962

If a sphere moves in a friction-producing fluid, propelled by a force F, which is constant in magnitude and direction, the velocity in the stationary terminal state will have the direction of the force and will be proportional to it. Therefore, the equation

$$(1) \qquad V = \gamma F$$

will be valid and γ could be designated the mobility in the liquid in question. The velocity will be characterized by the work performed by the acting force being equal to the frictional heat developed by the flow. According to equation (1)

$$(2) \qquad F \cdot V = \frac{V^2}{\gamma}$$

must, therefore, be the heat produced in the liquid in the unit of time.

Integrating the differential equations of the hydrodynamics that apply to this situation, Stokes[1] found that

$$(3) \qquad \gamma = \frac{1}{6\pi \mu a}$$

where a stands for the radius of the sphere, μ for the coefficient of friction. The hypothesis here is that $\frac{Va\, s'}{\mu}$ is a small figure as compared with the unit s' which designates the density of the liquid.

If the acting force, F, is the gravity, thus:

$$(4) \qquad F = \frac{4\pi}{3} a^3 g (s - s')$$

[1] Stokes, Cambr. Trans. 9, 1851 or Scientific Papers 3, p. 1, see also Lamb, Textbook of Hydrodynamics (Lehrbuch der Hydrodynamik), german by J. Friedel, Leipzig and Berlin, p. 682.

where s stands for the average specific gravity of the sphere. The radius of the sphere can be determined to be

$$(5)\qquad a = 3\sqrt{\frac{\mu V}{2g(s-s')}}$$

from the speed of the fall, the coefficient of friction being known, according to equations (1), (3), and (4).

This theory of Stokes has recently gained importance since it has been employed for the determination of atomic electrical charge. The principle of this test consists of letting a particle charged with the quantity of electricity e drop simply under the influence of gravity. The speed of fall V_1 can be calculated according to equations (1), (3), and (4) in the equation:

$$(6)\qquad \frac{4\pi}{3} a^3 g(s-s') = 6\pi\mu a V_1$$

and then overlaying the field of gravity by an equi- (or inverse-) directional electrical field of the intensity $\mathcal{E}$, of which a speed of fall will result, to be calculated from

$$(7)\qquad \frac{4\pi}{3} a^3 g(s-s') + e\mathcal{E} = 6\pi\mu a V_2$$

From (6) and (7) follows the radius of the particle as well as its electrical charge, namely,

$$(8)\quad a = 3\sqrt{\frac{\mu V_1}{2(s-s')g}} \qquad e\mathcal{E} = 9\pi(V_2 - V_1)\sqrt{\frac{2\mu^3 V_1}{(s-s')g}}$$

At the Physicists' meeting in Koenigsberg last year, Mr. Ehrenaft[1] reported on his determination of the atomic electrical charge, and it appeared to result from his tests that the atomic charge is not constant but that some particles show a higher, some

[1] Ehrenhaft, Phys. Zs. 11, 1910, p. 940.

a lower charge, as it also has been shown by other methods.

The discussion resulting from this report centered around the question as to what would be the reasons for the fluctuations of the results and thus, among other things, the premises of Stoke's theory was discussed.

First of all, it was indicated by Mr. Sommerfeld[2] that one of the pre-conditions for the use of Stoke's formula is the spherical shape of the falling particles and that disc- or rod-shaped particles would compe up with too small values for e, using the formula which is valid for a sphere.

During a private conversation with some colleagues after the session described above, I posed the question:"Now how is it exactly that discs or rods fall in a friction-producing fluid?" and one person answered: "discs drop down edgewise". He gave as a reason for his statement that each body would fall in such a way as to adapt itself to the least resistance of the surrounding fluid. Others, however, gave the opinion: " No. Discs fall front-wise". And, to make their point, pointed out that playing cards drop frontwise without any disturbance. When they are dropped edgewise, they turn end over end several times before reaching the ground. These are phenomena which are well-known in aviation. Finally, some one still mentioned that he recalls from his boyhood having dived for plates which always sunk right side up, as they stand on the table, that is, frontwise.

Since it appears quite evident that the opinions as to how a plate or rod fall in a friction-producing liquid, are quite divided, may I be allowed to deal with problem in the following lines,

[2] A. Sommerfeld, Phys. Zs. 11, 1910, p. 949.

entirely apart from the discussion of Ehrenhaft's test which, in the meantime, has gone a completely different way.

First of all, let it be noted that a minimal principle of the resistance in the sense used above for the movement of a solid body in a liquid does not exist. What seems to have happened here is a confusion with a theorem by Helmholz[1] which states that under given surface conditions, the development of heat at a stationary flow is lower than at any other movements of the liquid. Furthermore, the experiment dealing with dropping of playing cards proves nothing with regards to our problem as we are dealing only with the border-line case of infinitely low speeds treated by Stokes in which the factors dependent on the inertia of the liquid can be disregarded. This is the case if $\frac{V\ell s'}{\mu}$ is small in comparison to 1. ℓ stands for a linear dimension of the falling body. Now, this condition was not fulfilled in any way during the tests since for air, s' is 0.00129 and μ is 0.00017. V would have to be still small in comparison to 0.01 cm per sec.

Since the tests which have been carried out are not relevant to the question that has been posed by us and since it would also be difficult to design any decisive experiments, we shall now try to determine what kind of solution is to be found in theory.

Instead of determining the velocity and the pressure existing in any place of the liquid while imparting a movement of a constant speed to the body present in this liquid, we can ask ourselves how a constant field of flow will be modified by a solid body resint in the same, and what forces and tortional moments must be brought to

[1] H. v. Helmholtz, Wiss. Abh. 1, p. 223.

bear for the purpose of holding the body static in this flow.

This problem is covered by these differential equations:

$$\mu \Delta v_x = \frac{\partial p}{\partial x}$$

(9) $$\mu \Delta v_y = \frac{\partial p}{\partial y}$$

$$\mu \Delta v_z = \frac{\partial p}{\partial z}$$

(10) $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

if one the surface of the solid body

(11) $$v_x = 0; \quad v_y = 0; \quad v_z = 0$$

and in infinity possibly the conditions

(12a) $$v_x = U; \quad v_y = 0; \quad v_z = 0$$

or

(12b) $$v_x = 0; \quad v_y = V; \quad v_z = 0$$

or

(12c) $$v_x = 0; \quad v_y = 0; \quad v_z = W$$

are valid.

The forces of pressure are given further by the formulae[1]:

(13)
$$p_{xx} = -p + 2\mu \frac{\partial v_x}{\partial x}; \quad p_{yz} = p_{zy} = \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)$$

$$p_{yy} = -p + 2\mu \frac{\partial v_y}{\partial y}; \quad p_{zx} = p_{xz} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

$$p_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}; \quad p_{xy} = p_{yx} = \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right)$$

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[1] H. Lamb, l, c, p. 562, par. 314.

The differential equations, the limit conditions, and the pressure are, as can be seen from equations (9) and(13), linear in the components of velocity, i.e., the superposition of two solution systems (part motions) results again in the possible solution system (resulting motion).

Consequently, the following principle is directly acceptable:

The components of the force and the tortional motion necessary for holding the solid body in a static position in the resulting motion are simply the summation of the corresponding components required for the part motions.

In particular: if a body with three planes of symmetry, that are vertical with respect to each other, whose lines of interjection, the "major axes", may lie parallel with the axes of the coordinates, is in the liquid, and if the velocity in the infinite has the components U, V, W, the solution is composed of the three part motions that result if we introduce as a condition in the infinite first equation (12a), second (12b), and third (12c).

Now in none of these part motions, characterized by the body lying parallel with one of its major axes to the undisturbed field of flow, any rotational moment is required, for reasons of symmetry, to hold the body in its position; therefore, in the resulting motion, in which the body has any orientation whatsoever in regard to the field of flow, no rotational moment is required either or, if we go back to the case in which the liquid is situated in infinity, the body, however, is moved in the same in a straight line, the principle valid is:

No tortional moment acts upon the body because of the forces of pressure of the liquid. Discs or rods which have three planes

of symmetry in right angles with respect to each other, do, therefore, show no tendency to act in any particular way when falling slowly in a liquid.

While the results up to this point were the consequence of the linearity of the equations and of the properties of symmetry of the body, we have to define the shape of the body more precisely in order to be able to determine the process of the motion in detail.

As a result, we will assume that a disc is a flattened-out rotational ellipsoid while a rod is an elongated rotational ellipsoid; the long half axis will designate <u>a</u>, the short one <u>c</u>. The formulae pertinent to the stationary motion have been established by Oberbeck[1]

From these formulae, it can be seen that the mobility γ, i.e. the relation of the velocity of the body to the acting force can be calculated from the

$$\gamma_a = \frac{1}{6\pi\mu a}\,\delta_a \tag{14}$$

or

$$\gamma_c = \frac{1}{6\pi\mu a}\,\delta_c \tag{15}$$

depending whether the motion occurs in the direction of the long or the short axis.

In this case[2], for discs where $a = b > c$

$$\delta_a = a\,\frac{3}{8}\int_0^\infty \frac{d\lambda}{\sqrt{c^2+\lambda}}\left(\frac{1}{a^2+\lambda}+\frac{a^2}{(a^2+\lambda)^2}\right) \tag{16}$$

$$\delta_c = a\,\frac{3}{8}\int_0^\infty \frac{d\lambda}{\sqrt{c^2+\lambda}}\left(\frac{1}{a^2+\lambda}+\frac{c^2}{(a^2+\lambda)(c^2+\lambda)}\right) \tag{17}$$

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1 Oberbeck, Crelles Journal 81, 1876, p. 62; see also H. Lamb, l.c. par 326.

2 H. Lamb, l.c., par 326, formulae (6), (7), (14), (15).

and for rods, where a > b = c:

$$(18) \quad \delta_a = a \frac{3}{8} \int_0^\infty \frac{d\lambda}{\sqrt{a^2+\lambda}} \left(\frac{1}{c^2+\lambda} + \frac{a^2}{(c^2+\lambda)(a^2+\lambda)} \right)$$

$$(19) \quad \delta_c = a \frac{3}{8} \int_0^\infty \frac{d\lambda}{\sqrt{a^2+\lambda}} \left(\frac{1}{c^2+\lambda} + \frac{c^2}{(c^2+\lambda)^2} \right)$$

Carrying out the squares for discs gives:

$$(16') \quad \delta_a = \frac{3}{4} \left[\frac{1 - 2\frac{c^2}{a^2}}{\sqrt{1-\frac{c^2}{a^2}}} \operatorname{arc\,cos} \frac{c}{a} + \frac{c/a}{1-\frac{c^2}{a^2}} \right]$$

$$(17') \quad \delta_c = \frac{3}{8} \left[\frac{3 - 2\frac{c^2}{a^2}}{\sqrt{1-\frac{c^2}{a^2}}} \operatorname{arc\,cos} \frac{c}{a} - \frac{c/a}{1-\frac{c^2}{a^2}} \right]$$

(16') is used for edgewise motion while (17') is used for frontwise motion.

If the disc is flattened to the point where the second powers of $\frac{c}{a}$ can be neglected, (18') and (17') were transform into:

$$(16'') \quad \delta_a = \frac{9\pi}{16} \left(1 - \frac{8}{3\pi} \frac{c}{a} \right)$$

$$(17'') \quad \delta_c = \frac{3\pi}{8}$$

If, on theeother hand, the flattened rotational ellipsoid deviates so little from the spherical form that $1 - \frac{c}{a}$ becomes very small the equation will become:

$$(16''') \quad \delta_a = 1 + \frac{2}{5} \left(1 - \frac{c}{a} \right)$$

$$(17''') \quad \delta_c = 1 + \frac{1}{5} \left(1 - \frac{c}{a} \right)$$

In the case of rods, the calculation will be quite similar. Carrying out the squares required in (18) and (19) will yield:

$$(18') \quad \delta_a = \frac{3}{8}\left[\frac{1-\frac{c^2}{a^2}}{\sqrt{1-\frac{c^2}{a^2}}}\, \mathrm{Lg}\, \frac{1+\sqrt{1-\frac{c^2}{a^2}}}{1-\sqrt{1-\frac{c^2}{a^2}}} - \frac{2}{1-\frac{c^2}{a^2}}\right]$$

$$(19') \quad \delta_c = \frac{3}{16}\left[\frac{2-3\frac{c^2}{a^2}}{\sqrt{1-\frac{c^2}{a^2}}}\, \mathrm{Lg}\, \frac{1+\sqrt{1-\frac{c^2}{a^2}}}{1-\sqrt{1-\frac{c^2}{a^2}}} + \frac{2}{1-\frac{c^2}{a^2}}\right]$$

(18') and (19') are used for the motions parallel with and vertical to the axis of the rod respectively.

For small values of $\frac{c}{a}$:

$$(18'') \quad \delta_a = \frac{3}{4}\left\{2\,\mathrm{Lg}\,\frac{2a}{c} - 1\right\}$$

$$(19'') \quad \delta_c = \frac{3}{8}\left\{2\,\mathrm{Lg}\,\frac{2a}{c} + 1\right\}$$

For small values of $1 - \frac{c}{a}$:

$$(18''') \quad \delta_a = 1 + \frac{4}{5}\left(1 - \frac{c}{a}\right)$$

$$(19''') \quad \delta_c = 1 + \frac{3}{5}\left(1 - \frac{c}{a}\right)$$

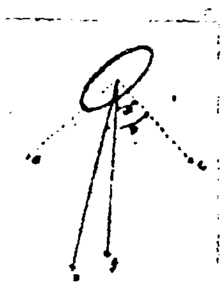
In Table 1, the values of δ_a and δ_c have been calculated for various axis ratios according to equations (16'), (17'), and (18') and (19') for discs and rods.

TABLE 1

$\frac{c}{a}$	for discs		for rods	
0.0	1.767	1.178		
0.1	1.631	1.174	3.763	2.617
0.2	1.516	1.160	2.799	2.108
0.3	1.418	1.145	2.267	2.815
0.4	1.335	1.125	1.914	1.606
0.5	1.261	1.105	1.662	1.451
0.6	1.196	1.084	1.467	1.326
0.7	1.139	1.063	1.314	1.224
0.8	1.088	1.041	1.191	1.138
0.9	1.041	1.019	1.091	1.063
1.0	1.000	1.000	1.000	1.000

If the force, $\mathfrak{F}$, forms the angle ϑ with the disc (or rod) axis, the velocity vector v will produce an angle φ with the axis (the illustration relates to discs), the relationships being:

$$v_a = r_a \mathfrak{F}_a \qquad v_c = r_c \mathfrak{F}_c \tag{20}$$



$$v \sin\varphi = r_a F \sin\vartheta \qquad v \cos\varphi = r_c F \cos\vartheta \tag{20'}$$

By division of both equations, (20') gives:

$$tg\,\varphi = \frac{r_a}{r_c}\, tg\,\vartheta = \frac{\delta_a}{\delta_c}\, tg\,\vartheta \tag{21}$$

As a consequence, it follows that when $\vartheta = 0$ and $\vartheta = \frac{\pi}{2}$, the direction of the velocity and that of the force coincide.

The maximum deviation between the directions of velocity and for occur, according to (21) when

(21a) $$\varphi - \vartheta = \mathrm{Arc\,tg}\left(\frac{\delta_a}{\delta_c}\, tg\,\vartheta\right) - \vartheta$$

is a maximum. In the case of rods, $\vartheta - \varphi$ is positive and then

the relationship becomes:

$$\text{(22)} \qquad tg\,\vartheta = \sqrt{\frac{\delta_c}{\delta_a}}, \qquad tg\,\varphi = \sqrt{\frac{\delta_a}{\delta_c}}$$

that is, ϑ and φ complete each other to become 90^0 (for rods, ϑ and φ are interchangable).

In table 2, the angles ϑ in different axial ratios $\frac{c}{a}$ for which $\varphi - \vartheta$ for discs and $\vartheta - \varphi$ for rods, have the maximal value, and these values $\varphi - \vartheta$ and $\vartheta - \varphi$, or in other words, the deviations between the velocity and force directions, are shown.

Table 2

$\frac{c}{a}$	for discs ϑ	for discs $\varphi - \vartheta$	for rods ϑ	for rods $\vartheta - \varphi$
0.0	39^0 41'	11^0 32'	54^0 44'	19^0 28'
0.1	40 18'	9 24'	50 11'	10 22'
0.2	41 11'	7 38'	49 03'	8 06'
0.3	41 56'	6 08'	48 11'	6 22'
0.4	42 33'	4 54'	47 31'	5 02'
0.5	43 06'	3 48'	46 57'	3 54'
0.6	43 35'	2 50'	46 27'	2 54'
0.7	44 01'	1 58'	46 01'	2 02'
0.8	44 22'	1 16'	45 39'	1 18'
0.9	44 41'	0 38'	45 22'	0 44'
1.0	45 00	0 00	45 00	0 00

In Table 3, ultimately $\varphi - \vartheta$ and $\vartheta - \varphi$ are calculated as functions of ϑ for very flat discs and very thin rods ($\frac{c}{a} = 0$) for which $\frac{\delta_a}{\delta_c}$ is 1.5 (2 resp.).

Table 3

ϑ	for discs $\varphi - \vartheta$	for rods $\vartheta - \varphi$
0^0	0^0 00'	0^0 00'
10	4 49	4 58
20	8 38	9 41
30	10 54	13 54
40	11 32	17 14
50	10 47	19 22
60	8 57	19 06
70	6 21	16 04
80	3 18	9 26
90	0 00	0 00

From Table 2, it can be seen that from the non-vertical fall of particles in a liquid, conclusion can be drawn concerning their deviation from spherical shape, however, this criterion is not very sharply defined as was pointed out by Mr. Sommerfeld during an exchange of letters concerning this question. The reason for this is that, in the most extreme cases, the deviation from the vertical is $11^\circ\ 32'$ in the case of discs, and $19^\circ\ 28'$ in the case of rods.

The equations which are pertinent to the determination of the atomic charge (equations (6) and (7)) now take the form:

$$(6') \qquad \frac{4\pi}{3} a^2 c g (s-s') = \frac{6\pi\mu a V_1}{\delta}$$

$$(7') \qquad \frac{4\pi}{3} a^2 c g (s-s') + e\mathcal{E} = \frac{6\pi\mu a V_2}{\delta}$$

and from them:

$$a = 3\sqrt{\frac{\mu V_1}{2(s-s')g}}\ \frac{1}{\sqrt{\delta \frac{c}{a}}}$$

$$(8') \qquad e\mathcal{E} = 9\pi (V_2-V_1)\sqrt{\frac{2\mu^3 V_1}{(s-s')g}}\ \frac{1}{\sqrt{\delta^3 \frac{c}{a}}}$$

in place of equation (8) and, to be sure, the value δ is replaced by δ_1 and δ_2 according to whether the disc is falling edgewise or frontwise.

In the case of rods, the analogous equations are:

$$(8'') \qquad a = 3\sqrt{\frac{\mu V_1}{2(s-s')g}}\ \frac{1}{\sqrt{\delta \frac{c^2}{a^2}}}$$

$$e\mathcal{E} = 9\pi (V_2-V_1)\sqrt{\frac{2\mu^3 V_1}{(s-s')g}}\ \frac{1}{\sqrt{\delta^3 \frac{c^2}{a^2}}}$$

if in the these equations, the δ values valid for rods are substituted.

Sommerfeld mentioned in the discussion that the values calculated by Ehrenhaft for e would have to be multiplied by $\sqrt{\frac{a}{c}}$ in the particles were very thin discs or by $\frac{a/c}{\lg \frac{2a}{c}}$ if they were very thin rods. As we can see from formulae (8') and (8'') as well as from the assymtotic values of δ (equations (16''), (18''), and (19'')), this result is correct, no matter how the particles are orientated with respect to the vertical.